Ring Theory 8

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Dummit Foote Q33:—

Suppose $M_c = \langle a_1, a_2, \ldots, a_n \rangle$

c) If $\langle x-c \rangle = M_c$ then $|x-c| \in M_c$

$|x-c| = f(x)(x-c)$ $\quad f(x) = \frac{|x-c|}{x-c} \in \left\{ \begin{matrix} -1 \\ 1 \end{matrix} \right.$ but not possible

d) Let $f = |a_1| + |a_2| + \ldots + |a_n|$

Then f is cont. in $[0,1]$

$f \in M_c$

Let $g = \sqrt{f}$ $\quad \because [f \geq 0]$

$g = r_1 a_1 + r_2 a_2 + \ldots + r_3 a_3 \leq |r_1||a_1| + \ldots + |r_n||a_n| \leq rf$ $\quad \because r \in R$

where $r = |r_1| + |r_2| + \ldots + |r_n|$

$\exists\ b \neq c$ such that $a_i(b) \neq 0\ \forall i$ ... [as M_c will have non-zero functions which are non-zero at $b \neq c$]

This statement about b will be true for all b's $\neq c$

$\Rightarrow \forall b \neq c \quad a_i(b) \neq 0 \ \forall i$.

$g = \sqrt{f} \leq rf \quad \Rightarrow \quad r \geq \frac{1}{\sqrt{f}}$ ... (not including c in domain)

$\Rightarrow \lim_{x \to c} r \geq \lim_{x \to c} \frac{1}{\sqrt{f}} \Rightarrow \lim_{x \to c} r = \infty \quad \Rightarrow r$ does not exist as $r \in R$

$\Rightarrow \Leftarrow$

So M_c is not finitely generated

Q) $\forall a \in R$ (non-zero comm. ring). At least one of a or $1-a$ is a unit. Prove that R is a local ring

Ans:— Let M to be a maximal ideal of R

Let x be a non-unit of R and $x \notin M$

M is a subset of ideal generated by $\langle M, x \rangle$

Then ideal generated by $\langle M, x \rangle = I$ should be R

$1 = m_1 + r_1 x$ ... $m_1 \in M$ & $r_1 \in R$

$m_1 = 1 - r_1 x$ $\qquad 0 = m_2 + r_1 x$ ... $m_2 \in M$

$1 = m_1 + r_1 x$

$m_1 = 1 - r_1 x$ $\qquad$ $0 = m_2 + r_1 x \quad \dots m_2 \in M$

$1 \notin M \;, 0 \in M$ $\qquad$ $\Rightarrow m_2 = -r_1 x$

(To be done later)

Q) Let A be an integral domain and I be an ideal of A. Is A/I is an integral domain.

Ans:- Let A/I is integral domain. Then $\exists$ $a, b \in R$ such that $(a+I)(b+I) = 0+I$

$\Rightarrow a \in I$ or $b \in I$ $\Rightarrow$ I is prime.

$\Rightarrow a+I = 0+I$ or $b+I = 0+I$

Let I be prime, then, let $ab \in I$ for $a, b \in R$

$\Rightarrow ab \in I \Rightarrow ab + I = 0 + I$

$\Rightarrow ab + I = (a+I)(b+I) = 0+I$

$\Rightarrow a \in I$ or $b \in I$

$\Rightarrow a+I = 0+I$ or $b+I = 0+I$

$\Rightarrow A/I$ is integral domain

Q) Let A be a finite ring with $1 \in A$. Prove that every element in A is either a unit or a zero divisor.

Ans:- Let $A = \{a_1, a_2, \dots, a_n\}$, $a_1 = 1$

Let $a_i \in A$ such that $a_i \neq 0$ and $a_i \neq 1$

$a_i, a_i^2, a_i^3, \dots, a_i^{n+1}$ $\Rightarrow a_i^m = a_i^k$ for some $m > k$ where $1 \leq m, k \leq n+1$

$\Rightarrow a_i^m - a_i^k = 0$

$\Rightarrow a_i^k (a_i^{m-k} - 1) = 0$

If $a_i^{m-k} - 1 = 0$ $\Rightarrow a_i^{m-k} = 1 \Rightarrow a_i$ is a unit

If $a_i^{m-k} - 1 \neq 0$, then, let $l \leq k$ be the least positive integer

If $a_i^{m-k} - 1 = 0 \Rightarrow a_i \quad -1 \rightarrow$ "

If $a_i^{m-k} - 1 \neq 0$, then, let $l \leq k$ be the least positive integer such that $a_i^l(a_i^{m-k} - 1) = 0$, $l \leq k$

Also as $a_i \neq 0$ we get $l > 1 \Rightarrow a_i \underbrace{a_i^{l-1}(a_i^{m-k} - 1)}_{\neq 0} = 0$

$\Rightarrow a_i$ is a zero divisor